THE GREAT PYRAMID OF GIZA

(Some Elegant Numerical Relationships)

Some time ago I devoted a few hours to playing with the dimensions of the Great Pyramid at Giza as an experiment with numbers. Some of the relationships that are mentioned below are quite famous, such as the ones involving the numbers ***&pi*** and ***&phi***, but many of the others have probably not been pointed out before. I am not suggesting that these relationships are intentional, or that the architect who designed the Great Pyramid was even aware of them. Quite the contrary, this experiment shows that such relationships are rather easy to find. Most Egyptologists would discount the kinds of relationships exhibited below as merely unintentional coincidences, elegant as they might seem, unless they could be backed up by other substantial evidence concerning the intentions or the methods of the architect. The experiment would seem to amply justify that skeptical point of view. A similar point is made in another experiment, described on the page Secrets of a Certain Pentagon, where I searched for relationships which are elegant as well as highly accurate.

Some scholars have investigated the question of what the intentions of the architects involved in building the various pyramids might have been. For example, several pages in an article ***Mathematical Bases of Ancient Egyptian Architecture and Graphic Art*** by G. Robins and C. Shute (Historia Mathematica, May, 1985) are devoted to this very question. The discussion in that article is based mostly on the Rhind Papyrus together with their own measurements of the pyramids. The Rhind Papyrus is mostly a series of mathematical exercises with solutions. Some of those exercises concern pyramids and represent the slant-angle of the faces of a pyramid by a quantity called the "seked." (It is really the inverse of the slope of the slant-angle and is measured by the number of palms and fingers of horizontal distance per cubit of vertical distance.) The conclusion that those authors come to is that the architects of the pyramids simply chose the seked to be five palms + 0, 1, 2, or 3 fingers per cubit. For the Great Pyramid, the choice was five palms, two fingers per cubit. This amounts to choosing the slope of each face to be 14/11, which is exactly the numerical relationship #2 below. Other scholars have made a similar argument. However, as I will mention at the end, I have come to believe that the choice was not quite that simple and that the Egyptian architect who designed the Great Pyramid had in mind not just relationship #2, but also #3.

According to Mysteries of the Great Pyramids by A. Pochan, the angle that each face of the Pyramid makes with the base is approximately 51.85^{o}. To make the numbers precise in the following experiment, I decided to consider a mathematically perfect pyramid with a square base, isosceles triangles as faces, and with 51.85^{o} as the exact angle which each face makes with the base. I then searched for numerical relationships involving the following six lengths incorporated in this pyramid.

s = the length of one side of the base
p = the perimeter of the base
d = the length of a diagonal of the base
h = the height of the pyramid
r = the length of one edge (or ridge) of the pyramid
f = the length of the line joining the midpoint of
one side of the base to the apex.

To make this clear, each of these quantities represents a distance. If you stand at one corner of the base of the pyramid, s is the distance from you to the next corner, d is the distance from you to the opposite corner, and r is the distance from you to the apex of the pyramid. If you stand at the base of the pyramid halfway between two corners, then f is the distance from you to the apex of the pyramid. If you stand inside the pyramid at the very center, then h is the distance from you to the apex, which is of course just the total height of the pyramid. The perimeter p of the pyramid is the total distance completely around the base: p = s+s+s+s = 4s. The diagonal d is related to the side s by $d = \sqrt{2}\, s$. If we take the length s of one side of the base as the unit of measurement, then by using some trigonometry we obtain the following values (to six decimal places):

s = 1 , p = 4 , d = 1.414213..., h = .636528..., r = .951403..., f = .809425....

Below are some of the relationships which are exhibited by the pyramid. Relationships 1, 2, 3 and 5 are discussed in Pochan's book mentioned above. They are also discussed in Peter Tompkin's book Secrets of the Great Pyramid, especially the very famous relationships involving the numbers ***&pi*** and ***&phi*** (which are 1 and 5 below). We will use the following notation to indicate approximations: If a and b are two positive numbers , we will write $a \sim b$ to mean that a and b are approximately equal . We will put in parentheses the error of the approximation, which is the difference between a and b expressed as a rough percentage of the number b. For example, if we write $199 \sim 200$, then the difference is 1 and the error is .5% because 1 is .5% of 200. All of the relationships that we will describe involve only the "shape" of the pyramid that we are considering. We won't discuss at all the question of the actual size or the orientation of the Great Pyramid at Giza, although these issues have also been a popular subject of speculation.

1. p / h ~ 2***&pi*** (.015%)

 Here *pi* = 3.141592.... For more about the number ***&pi***, visit the The *Pi* Pages. This relationship means that the perimeter of the Great Pyramid is approximately equal to the circumference of a circle of radius h.

2. p / h ~ 2(22/7) (.026%)

 This relationship is related to the first one because 22/7 is a famous classical approximation to ***&pi***. In fact, the ratio p/h is somewhat larger than 2***&pi***and somewhat smaller than 2(22/7). An equivalent way to state this relationship is in terms of the slope of each face of the Pyramid: h /(.5s)=14/11. If you start from the center of one side and climb directly toward the apex of the Pyramid, then for every 14 feet higher (vertically) above the base you get, you will move approximately 11 feet closer (horizontally) to the center.

3. h /(.5d) ~ 9/10 (.02%)

 The ratio h/(.5d) is the slope of each edge of the Pyramid. This relationship means that if you are climbing up one of the edges of the Pyramid, then for every 9 feet higher above the base that you climb, you will move approximately 10 feet closer to the center.
 The inverse of this slope is 10/9=1+1/9.

4. f /(.5s) ~ F_9 /F_8 (.015%)

 Here F_8 = 21 and F_9 = 34 are the 8th and 9th Fibonacci numbers.

5. f /(.5s) ~ ***&phi*** (.05%)

Here ***&phi*** = $(1+\sqrt{5})/2$ = 1.618033..... is the famous Golden Mean. One should compare this relationship to the more accurate relationship #4. It can be proved that ***&phi*** can be approximated extremely accurately by the ratios of consecutive Fibonacci numbers. In fact, in a certain precise sense, the best approximations to the irrational number ***&phi*** by rational numbers are given by the rational numbers of the form F_{n+1}/F_n, where F_n is the n-th Fibonacci number and F_{n+1} is the (n+1)-st Fibonacci number. It just so happens that f/(.5s) is fairly close to ***&phi***, but even closer to the nearby rational number F_9/F_8.

6. $r / f \sim e^2 / 2$***&pi*** (.06%)

Here e = 2.718281.... is an extremely important constant which was introduced into mathematics in the 18th century. For various ways to define e, visit here.

7. $s / r \sim 1 + 1/3^3 + 1/5^3 + 1/7^3 + 1/9^3 + \ldots\ldots$ (.07%)

The infinite series which is on the right-hand side of this relationship converges to a certain number, but mathematicians have been able to prove very little about this number. It probably cannot be expressed in any simple way in terms of ***&pi***, e, or other well-known numbers. It is closely related to Apery's constant. In the late 1970's , a proof of the irrationality of this number was discovered by the French mathematician Roger Apery.

8. $3s / h \sim e /$***&gamma*** (.08%)

Here ***&gamma*** = .577215.... is the "Euler-Mascheroni constant." The ratio 3s/h can also be written as $s^3/{}^1/_3hs^2$, which is the ratio of the volume of a cube built on the base of the pyramid to the volume of the pyramid itself.

9. $s / h \sim e / \sqrt{3}$ (.1%)

10. $h / f \sim$ ***&pi*** $/ 4$ (.12%)

11. $s / f \sim 1 + 1/3^2 + 1/5^2 + 1/7^2 + 1/9^2 + \ldots..$ (.14%)

This infinite series is known to converge to the number ***&pi***$^2/ 8$.

COMMENTARY: We should emphasize that the relationships listed above are for the perfect pyramid where the base makes an angle of exactly 51.85^o with each face. The actual Great Pyramid at Giza is not perfect. The base is not precisely a square. According to the 1925 survey carried out by J.H. Cole, the angles at the four corners and the lengths of the four sides of the base of the Great Pyramid have the following measurements:

NorthWest	89°59'58"
NorthEast	90°3'2"
SouthEast	89°56'27"
SouthWest	90°0'33"

North Side	230.253 meters
South Side	230.454 meters
East Side	230.391 meters
West Side	230.357 meters

As these measurements show, the builders of the Great Pyramid achieved a rather impressive level of accuracy, assuming that their intention was to make the base a perfect square. The average length of the four sides is 230.364 meters and the greatest discrepancy from that average is .111 meters (the North Side), which is just 4 inches. The discrepancy between the longest and shortest sides is just 8 inches. As a percentage of the length, this is less than .09%. As for the angles, the SouthWest angle is extremely close to 90^o, the percentage error being just .01%. The largest discrepancy from 90^o is .065% (for the angle at the SouthEast corner).

The Great Pyramid is not a perfect pyramid in other ways too. The apex is missing. The faces are slightly concave. As for the angle of 51.85^o which we used in the above experiment, it is accurate to within an error of about .03% according to Pochan. In Mark Lehner's authoritative book The Complete Pyramids, the angle that each face makes with the base is given as 51^o51'14" , which is just slightly more than 51.85^o . In any case, the approximations that we have given above will still be fairly accurate for the actual Pyramid, but the errors may differ somewhat from those given above. Some of the error estimates may turn out to be better, some worse.

But the most interesting question to ask is whether any of these relationships were intentional. That is, did the Egyptian architect(s) have one or more of these relationships in mind when designing the structure? For some of the relationships given above, the answer is rather clear. It is hard to believe that the Egyptians had any knowledge or awareness whatsoever of numbers like e , ***&gamma***, or the infinite series occurring in #7 and #11. These numbers entered mathematics during the 17th and 18th centuries. We have included them just to make the point that it is not so hard to find coincidental numerical relationships and their existence should not necessarily be regarded as significant. But for some of the other relationships, it seems difficult to answer the question with any confidence. Since there doesn't seem to be any written documentation which directly explains the design of the Great Pyramid, one has to attempt to be a mind-reader. I mentioned earlier the argument by Robins and Schute based on the Rhind Papyrus. I myself do not completely agree with their conclusion and would propose that both relationships #2 and #3 reflect the intentions of the architect. To be more precise, the architect may have chosen one of those relationships, cognizant of the fact that the other relationship would then be true. Both of these relationships may have been used in building the pyramid with accuracy. In the essay The Slopes of the Egyptian Pyramids, I will present arguments in support of that possibility. As a consequence, it would seem reasonable to believe that all of the other relationships listed above, including the famous relationships involving the numbers ***&pi*** and ***&phi***, are merely unintended, accidental consequences of this choice. One can find a more carefully discussion of the relationship involving ***&pi*** in my essay Pi and the Great Pyramid.

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